Math 4200
Monday September 14

1.5 harmonic functions, harmonic conjugates; 1.6 analtyic functions constructed via e^z and $\log z$. (We'll have more section 1.6 discussions on Wednesday.)

Announcements:

Review example Let $f(z) = \log z = \ln |z| + i\, arg(z)$. Prove $f(z)$ is analytic with $f'(z) = \frac{1}{z}$, away from $z = 0$ (for any continuous branch choice i.e. by specifying $arg(z)$ continuously in a neighborhood of z). Do this three ways! Each of these is easier than trying to verify the limit definition directly.

1) Inverse function theorem and chain rule.

2) Rectangular Cauchy-Riemann equations plus continuous partials, via the Cauchy-Riemann Theorem.

3) Polar coordinate CR equations, plus C^1. (You worked out the CR equations in polar coordinates in your last homework probably using 3220 chain rule; we can recover them quickly from the chain rule for curves, writing $f(z) = f\left(r\, e^{i\theta}\right)$.

1.5 Harmonic functions and harmonic conjugates.

Let $f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$ be analytic in an open domain A, and assume u, v have continuous first and second partial derivatives. (The shorthand for this is $u, v \in C^2(A)$.) Then from Cauchy Riemann

$$u_x = v_y$$
$$u_y = -v_x$$

we compute

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

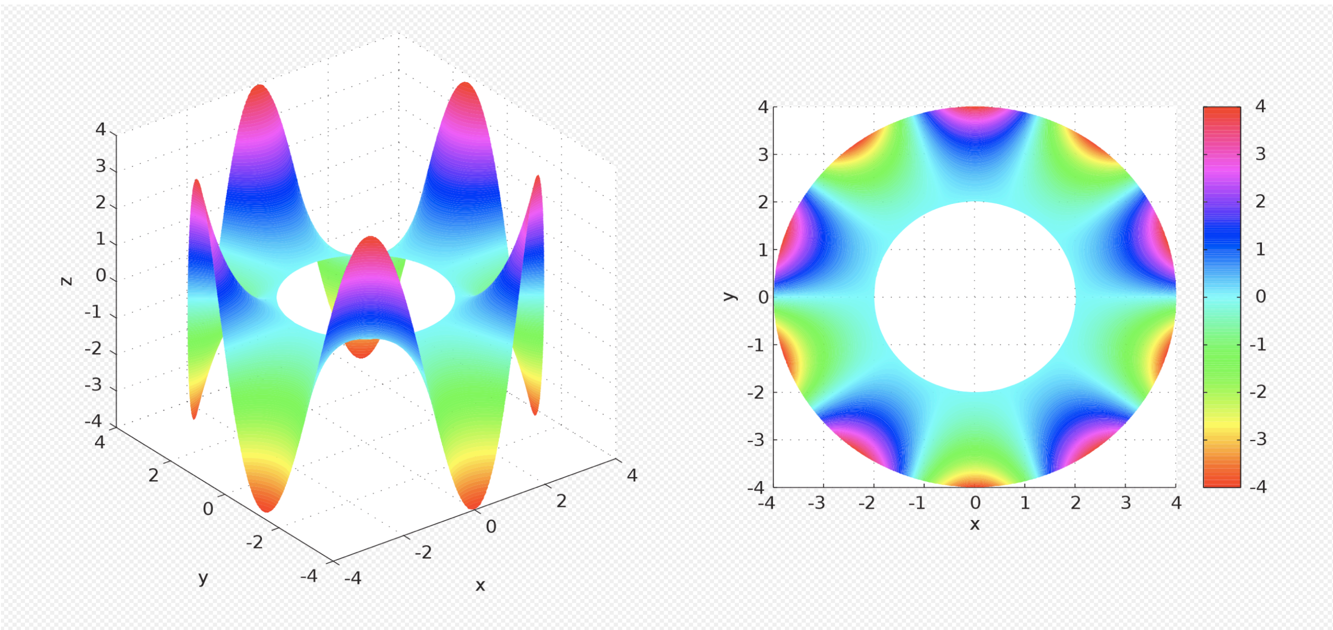
(Recall from 3220 or multivariable calculus that $v_{yx} = v_{xy}$ when all second partial derivatives are continuous.)

Def Let $U(x, y)$ be a C^2 function in a domain $A \subseteq \mathbb{R}^2$. Then U is *harmonic* in A if it satisfies the partial differential equation

$$U_{xx} + U_{yy} = 0.$$

Def The partial differential equation above is called *Laplace's equation*.

Harmonic functions are important in pure and applied math, as well as in physics. Also harmonic functions of three or more variables. If you've taken any class on partial differential equations or electro-magnetism, you've seen harmonic functions before. Here's the graph of a certain harmonic function defined on an annulus, taken from the Wikipedia page on harmonic functions. It could represent a the equilibrium temperature distribution on a thin metal plate, where the temperature values are specified as indicated on the inner and outer circles of the annulus.



Def Let $A \subseteq \mathbb{C}$ open, and let $u \in C^2(A)$ be a harmonic function. A function $v(x, y)$ such that

$$f(z) = f(x + i\, y) = u(x, y) + i\, v(x, y)$$

is analytic in A is called a *harmonic conjugate* to $u(x, y)$.

Theorem If $u(x, y) \in C^2(A)$ where A is an open *simply connected* domain. (A domain is called simply connected if its connected and "has no holes". We'll discuss this concept more carefully in the next chapter.) Then there exists a harmonic conjugate $v(x, y)$ to $u(x, y)$, unique up to an additive constant.

proof: $u \in C^2(A)$, $u_{xx} + u_{yy} = 0$ is given. The system for finding $v(x, y)$ has to be consistent with the Cauchy-Riemann equations for f:

$$v_x = P(x, y) \qquad \left(= -u_y\right)$$
$$v_y = Q(x, y) \qquad \left(= u_x\right)$$

When you studied *conservative vector fields* and *Green's Theorem* in multivariable calculus you learned that a vector field $[P, Q]^T$ is actually the gradient of a function $v(x, y)$ locally if and only if the necessary condition that $v_{x\,y}$ would equal $v_{y\,x}$ holds:

$$P_y = Q_x$$

In our case, since P, Q are partials of $u(x, y)$ this integrability condition reads as

$$-u_{y\,y} = u_{x\,x}$$

which holds since u is harmonic!

Example Let $u(x, y) = x\, y$. Show u is harmonic. Then find its harmonic conjugate $v(x, y)$ and identify the analytic function $f(z) = u(x, y) + i\, v(x, y)$.

Rigouous proof of key point for harmonic conjugate construction

Theorem Let A be an open simply connected domain in $\mathbb{R}^2$. Let $[P, Q]$ be a C^1 vector field defined on A. Then there is a function $v \in C^2(A)$ so that

$$v_x = P(x, y), \qquad v_y = Q(x, y)$$

if and only if the curl of the vector field is zero:

$$P_y = Q_x.$$

This condition is necessary since if v exists then $v_{x\,y} = P_y$ and $v_{y\,x} = Q_x$.

Local proof: (Once we've carefully defined simply-connected domains in Chapter 2, the global theorem in a simply connected region follows from this local version.) Let P, Q be real differentiable, with continuous partials in $B_r((x_0, y_0))$, $r > 0$, and satisfying the "zero curl" condition $P_y = Q_x$. Let $v(x_0, y_0)$ be any chosen constant. Then $\forall$ points $(x_1, y_1) \in B_r((x_0, y_0))$ define $v(x, y)$ in a way which would be consistent with $P = v_x$, $Q = v_y$ if we already knew the function $v(x, y)$. There are two ways to do this using the fundamental theorem of Calculus, and following sides of a rectangle. The curl condition ensures that both routes yield the same value:

(1) $$v(x_1, y_1) = v(x_0, y_0) + \int_{x_0}^{x_1} P(x, y_0)\, \mathrm{d}x + \int_{y_0}^{y_1} Q(x_1, y)\, \mathrm{d}y$$

(2) $$v(x_1, y_1) = v(x_0, y_0) + \int_{y_0}^{y_1} Q(x_0, y)\, \mathrm{d}y + \int_{x_0}^{x_1} P(x, y_1)\, \mathrm{d}x$$

The two formulas agree iff the difference of their right hand sides equals zero:

$$\int_{x_0}^{x_1} P(x, y_0) - P(x, y_1)\, \mathrm{d}x + \int_{y_0}^{y_1} Q(x_1, y) - Q(x_0, y)\, \mathrm{d}y = 0$$

iff

$$\int_{x_0}^{x_1} \left(-\int_{y_0}^{y_1} P_y(x, y)\, \mathrm{d}y \right) \mathrm{d}x + \int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} Q_x(x, y)\, \mathrm{d}x \right) \mathrm{d}y = 0.$$

This last equality holds because $-P_y + Q_x = 0$ in the rectangle.

Finally, using (1) and FTC to compute v_{y_1} we see $v_{y_1}(x_1, y_1) = Q(x_1, y_1)$; and using (2) we compute $u_{x_1}(x_1, y_1) = P(x_1, y_1)$ QED.

1.6 The zoo of basic analytic functions, their derivatives, and branches for their inverses. (We'll continue section 1.6 on Monday.)

Def If $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$, then f is called *entire.*

Examples:

$f(z) = z^n,\ n \in \mathbb{Z} \setminus \{0\}$ $\qquad f'(z) = n\, z^{n-1}$

$f(z) = e^z$ $\qquad f'(z) = e^z$

$f(z) = \cos(z) = \frac{1}{2}\left(e^{i z} + e^{-i z}\right)$ $\qquad f'(z) =$

$f(z) = \sin(z) = \frac{1}{2 i}\left(e^{i z} - e^{-i z}\right)$ $\qquad f'(z) =$

Here is a non-entire function, but you can define it as a differentiable function locally, or using any branch domain for $\log z$:

$f(z) = z^a := e^{a \log(z)}$, $a \in \mathbb{C}$ $\qquad\qquad$ $f'(z) =$

Question: For $f(z) = z^a$ as above, does the multi-value definition agree with $f(z) = z^n$, $n \in \mathbb{Z}$?

Question: For $f(z) = z^a$ as above, does the multi-value definition agree with the multivalue definition of the n^{th} root function $f(z) = z^{\frac{1}{n}}$, $n \in \mathbb{N}$?

Math 4200-001
Week 3 concepts and homework
1.5 - 1.6
Due Friday September 18 at 11:59 p.m.

1.5 25, 26, 27, 28, 31.
1.6 1c, 2abc, 3a, 4, 5.

extra credit (5 points) From class discussions we know that complex analytic f correspond to real-differentiable maps $F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ which have rotation-dilation differential matrices, so that the differential map preserves angles between tangent vectors, i.e. is *conformal.* It turns out that any differential map which preserves angles between tangent vectors, and which also preserves orientation must be a rotation dilation. Prove this.

Hints: For each tangent vector $\gamma'(t_0) = \vec{v} \in T_{(x_0, y_0)}\mathbb{R}^2$, and writing $F(x, y) = (u(x, y), v(x, y))$, the differential map is given by

$$dF_{(x_0, y_0)}(\vec{v}) = (F \circ \gamma)'(t_0)$$

and the multivariable chain rule says we can compute this by the formula which uses the differential (aka derivative or Jacobian) matrix:

$$dF_{(x_0, y_0)}(\vec{v}) = DF(x_0, y_0)\,\vec{v} = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Your job is to show that if this differential map preserves angles, and orientation then the matrix must be a rotation dilation matrix. A good way to get started is to note that

$$\angle\, dF(\vec{v}), dF(\vec{w}) = \angle\, \vec{v}, \vec{w} \qquad \forall\ \vec{v}, \vec{w} \in T_{(x_0, y_0)}\mathbb{R}^2$$

implies that the two columns of the derivative matrix must perpendicular, by the choice $\vec{v} = [1, 0]^T, \vec{w} = [0, 1]^T$. Then make use of the dot product formula you know for (unoriented) angles, for at least one other good choice of $\vec{v}, \vec{w}$, to deduce that the magnitudes of the two columns must agree.

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\ \|\vec{w}\|}.$$

Finally, use the fact that two ordered vectors $\vec{v}, \vec{w}$ are positively oriented means that the determinant of the matrix with columns $[\vec{v}\ \vec{w}]$ is positive. (Geometrically this means that the signed angle from $\vec{v}$ to $\vec{w}$ is between 0 and π.) The differential map is orientation preserving means that it transforms positively oriented vectors to positively oriented vectors. As an aside, using determininants is how you define positive orientation for n vectors in $\mathbb{R}^n$, as the right hand rule no longer makes any sense when $n > 3$.